Date: 15.05.2014
Teacher: Ezgi Çallı
Number of Students: 19
Grade Level: 11
Time Frame: 45 minutes

DETERMINANT

1. Goal(s)

- Students will be able to develop an understanding about the concept of determinants.

2A. Specific Objectives (measurable)

- Students will be able to define minors and cofactors of matrices.
- Students will be able to define the ad joint of a matrix and use ad joint to find the inverse of 2x2 and 3x3 matrices.
- Students will be able to find the solution of system of linear equations A.X = B, by using $X = A^{-1}B$

2B. Ministry of National Education (MoNE) Objectives

- Minör ve kofaktör kavramlarını açıklar, 1x1, 2x2 ve 3x3 türündeki matrislerin determinantını hesaplar ve determinantın özelliklerini belirtir
- Ek (adjoint) matrisi açıklar, 2x2 ve 3x3 türündeki matrislerin tersini ek matris yardımıyla bulur

3. Rationale

Determinants basically help us to describe the nature of solutions of linear equations. The determinant of a real matrix is just some real number, telling you about the invertibility of the matrix and hence telling you things about linear equations wrapped up in the matrix.

Finding the determinant of the 2 x 2 matrix that describes those two vectors is the same as finding the area of the parallelogram formed by them. Adding one more dimension and we can get a parallelepiped for the shape and the volume for the determinant.

4. Materials

- Two colors of board markers.
- PowerPoint presentation.
- Worksheets printouts.

5. Resources

- Ortaöğretim Matematik 11. Sınıf Ders Kitabı (MEB yayınları)
- Mathematics Higher Level for the IB Diploma (OXFORD)
- http://www.utdallas.edu/dept/abp/PDF_Files/LinearAlgebra_Folder/Cofactors.pdf
- https://people.richland.edu/james/lecture/m116/matrices/applications.html

6. Getting Ready for the Lesson (Preparation Information)

- Teacher prepares name cards for students and writes students' name on those cards before the lesson.

- Teacher prepares a checklist including student names.
- Teacher prepares a PowerPoint presentation.
- Teacher prepares Worksheets.

7. Prior Background Knowledge (Prerequisite Skills)

  - The students should previously have learned matrix operations and their properties.
  - The students should be able to define the inverse of a matrix.
  - The students should be able to find the determinant of a 2x2 or 3x3 matrix.

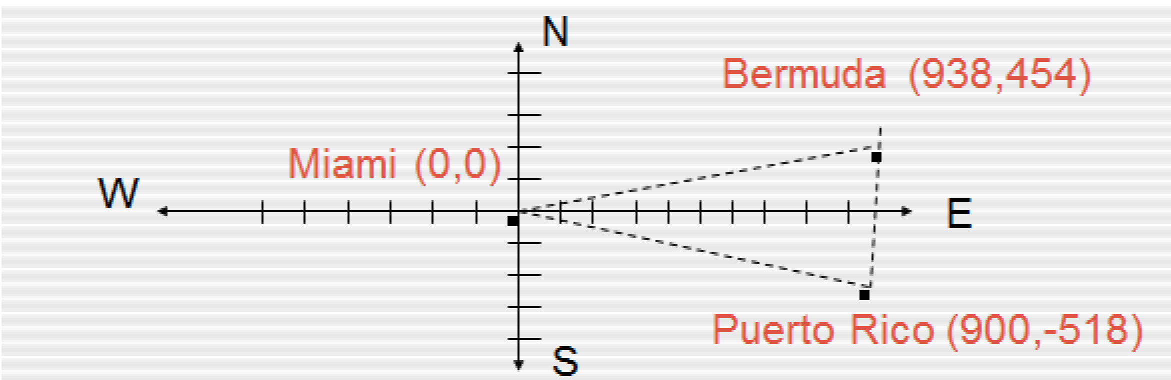
Lesson Procedures

Transition: Last lesson you learnt what a determinant is. What is a determinant, do you remember?

8A. Engage (10 minutes)

· The teacher reflects the opening problem to the board and asks the students if they know the Bermuda triangle.

The Bermuda Triangle is a large triangular region in the Atlantic Ocean. Many ships and airplanes have been lost in this region. The triangle is formed by imaginary lines connecting Bermuda, Puerto Rico, and Miami, Florida. Use a determinant to estimate the area of the Bermuda Triangle.



The approximate coordinates of the Bermuda Triangle's three vertices are: (938,454), (900,-518), and (0,0). So the area of the region is as follows:

$$Area = \pm\frac{1}{2}\begin{vmatrix} 938 & 454 & 1 \\ 900 & -518 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$Area = \pm\frac{1}{2}[(-458,884+0+0)-(0+0+408,600)]$$

$$Area = 447,242$$

Hence, area of the Bermuda Triangle is about 447,000 square miles.

Transition: Today we will learn some new concepts on determinant. Let's get familiar with them first.

B. Explore (10 minutes)

Steps to Finding Each Minor of a Matrix:

1. Delete the ith row and jth column of the matrix.
2. Compute the determinant of the remaining matrix after deleting the row and column of step 1.

Example: Find the minors of the matrix

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

*Note: This step procedure just outlines finding the minor M_{11} of the matrix.

1. Delete the ith row and jth column of the matrix.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

2. Compute the determinant of the remaining matrix after deleting the row and column of step 1.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \Rightarrow M_{11} = \det\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = (-1)(-1)-(1)(1) = 0$$

Using the same steps above, the other minors of the matrix are given below.

$$M_{12} = \det\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = -3 \qquad M_{13} = \det\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = 3$$

$$M_{21} = \det\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = -1 \qquad M_{22} = \det\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = -3$$

$$M_{23} = \det\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = 2 \qquad M_{31} = \det\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = 1$$

$$M_{32} = \det\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = -3 \qquad M_{33} = \det\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = 1$$

Thus, the minor matrix is given by

$$M = \begin{bmatrix} 0 & -3 & 3 \\ -1 & -3 & 2 \\ 1 & -3 & 1 \end{bmatrix}$$

Cofactors:

To find the cofactors of a matrix, just use the minors and apply the following formula:

$$\boldsymbol{C_{ij} = (-1)^{i+j} M_{ij}}$$

Where Mij is the minor in the ith row, jth position of the matrix.

Example: Find the cofactors of the matrix

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

We know that the minor matrix is given by

$$M = \begin{bmatrix} 0 & -3 & 3 \\ -1 & -3 & 2 \\ 1 & -3 & 1 \end{bmatrix}$$

So computing the cofactor matrix C yields

$$C = \begin{bmatrix} 0 & 3 & 3 \\ 1 & -3 & -2 \\ 1 & 3 & 1 \end{bmatrix}$$

Adjoint:

To find the adjoint of a matrix denoted by adj(A), just transpose the cofactor matrix.

Example: Find the adjoint of the matrix

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Since we know the cofactor matrix, we can just transpose the matrix which yields the following result.

$$adj(A) = C^T = \begin{bmatrix} 0 & 3 & 3 \\ 1 & -3 & -2 \\ 1 & 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 1 \\ 3 & -3 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

Transition: There are many useful applications of the determinant. Cofactor expansion is one technique in computing determinants.

C. Explain (5 minutes)

We discussed how these minors cofactors of a matrix are used to find the ad joint of A. Then by the ad joint and the determinant, we can develop a formula for finding the inverse of a matrix.

Finding Inverses Using the Ad-joint:

The inverse can be easily calculated using the following formula:

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Let's find the inverse of the matrix in the worksheet, by using the ad-joint.

Our matrix A

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$adj(A) = \begin{bmatrix} 0 & 1 & 1 \\ 3 & -3 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{bmatrix} = (1)(1)(3) = 3$$

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

$$= \frac{1}{3}\begin{bmatrix} 0 & 1 & 1 \\ 3 & -3 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1 & 1 & 1 \\ 1 & 1 & 1/3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1 & 1 & 1 \\ 1 & 1 & 1/3 \end{bmatrix}$$

Transition:

D. Extend (20 minutes)

After the explanation part, given examples will be solved:

Ex.1: Find the determinant of the given matrix $A = \begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}$ and compare it with $a_{11}.A_{11} + a_{12}.A_{12}$, $a_{21}.A_{21} + a_{22}.A_{22}$, $a_{11}.A_{11} + a_{21}.A_{21}$ and $a_{12}.A_{12} + a_{22}.A_{22}$. Represent your finding(s) with a few sentences.

Ex.2: Find the inverse of given matrix $A = \begin{bmatrix} 5 & 2 & 1 \\ 3 & 1 & -2 \\ -1 & -3 & 4 \end{bmatrix}$.

Ex.3: (IB Exam Question)

Find the values of a and b that the matrix A= $\begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}$ is the inverse of the matrix B= $\begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}$ by using adjoint matrix.

Ex.4: (IB Exam Question)

Given that A= $\begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$ and B= $\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ find X if $BX = A - AB$.

- Support students to analyze the question and develop their own approach by making meaningful connections. They can discuss their opinions with their desk mates.
- Give time to construct and present their ideas.

Transition: Thank you for this nice lesson, class.

E. Evaluate
The teachers used a checklist during the activity while monitoring students and took anecdotal notes.

9. Closure & Relevance for Future Learning
- Summarize what has been done briefly by asking the students.

10. Specific Key Questions:
Do you remember what determinant is? (comprehension)
How many minors does an m x m matrix have? (application)
What is the inverse of a matrix? (comprehension)
How can we find the inverse of a matrix? (synthesis)
11. Modifications
- If some students do well, use extra extension problems. While they were investigating, ask leading questions to them and want them to record their findings. Remember to use praises, appreciate, and encourage them.
- For struggling students, pay attention to be a more successful student academically to help them in the group work.