Class level: 9-10

Objectives:

<u>**IGCSE (0607-Extended, 2014):**</u>

(Extension to) 7.7. Linear inequalities on the Cartesian plane (shade unwanted regions)

<u>**MoNE:**</u>

(Extension to) 9.2.4.2. Denklem ve eşitsizlikleri gerçek/gerçekçi hayat durumlarını modellemede ve problem çözmede kullanır.

Materials: TI Calculator, MS Excel

Time: 45 minutes

Resource: Winston, W. L. (2004). *Operations research: Applications and Algorithms*. Belmont, CA: Thomson Learning.

Giapetto's Woodcarving Problem

Giapetto's Woodcarving, Inc., manufactures two types of wooden toys: soldiers and trains. A soldier sells for $27 and uses $10 worth of raw materials. Each soldier that is manufactured increases Giapetto's variable labor and overhead costs by $14. A train sells for $21 and uses $9 worth of raw materials. Each train built increases Giapetto's variable labor and overhead costs by $10. The manufacture of wooden soldiers and trains requires two types of skilled labor: carpentry and finishing. A soldier requires 2 hours of finishing labor and 1 hour of carpentry labor. A train requires 1 hour of finishing labor and 1 hour of carpentry labor. Each week, Giapetto can obtain all the needed raw material but only 100 finishing hours and 80 carpentry hours. Demand for trains is unlimited, but at most 40 soldiers are bought each week. Giapetto wants to maximize his weekly profit (Revenues - Costs). Formulate a mathematical model of Giapetto's situation that can be used to maximize Giapetto's weekly profit.

STEP 1: Understand the Problem

You need to help Giapetto in his decision making, by showing him a scientific approach. The scientific approach to decision making requires use of mathematical models. In order to do that, first you need to understand what your "objective" is.

a) **What is your objective in this question?**

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Next, you need to collect data about Giapetto's problem.

b) **What kinds of factors have an influence on Giapetto's weekly profit?**

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c) **Guess the optimal solution intuitively.**

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STEP 2: Formulate the Problem

The variables whose values should be controlled by the decision maker and influence the proximity to your objective are called *decision variables*.

a) **Define the decision variables**

    X=

    Y=

Now, you need to write a function by using the decision variables. The function to be minimized or maximized is called the *objective function*. Giapetto's objective is to choose X and Y to maximize the objective function.

b) **Write the objective function in terms of X and Y**

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{Hint}

    Weekly revenues

$$= (\$/soldier) * (soldier/week) + (\$/train) * (trains/week)$$

    Weekly raw material costs

$$= (\$/soldier) * (soldier/week) + (\$/train) * (trains/week)$$

    Weekly variable costs

$$= (\$/soldier) * (soldier/week) + (\$/train) * (trains/week)$$

Don't forget, there is always a tradeoff in economics. (A tradeoff is a situation that involves losing one quality or aspect of something in return for gaining another quality or aspect.) Now, the decision maker needs to determine what certain values of decision variables are possible or not. The restrictions on the values of decision variables are called *constraints*. Constraints can be expressed as algebraic inequalities.

c) **Determine the constraints in terms of X and Y**

Constraint 1: Each week, no more than 100 hours of finishing time may be used.

....

Constraint 2: Each week, no more than 80 hours of carpentry time may be used.

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Constraint 3: Because of limited demand, at most 40 soldiers should be produced each week.

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Constraints 4 & 5: Can the decision variables be negative?

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(Warning: Be careful about the units of the terms in a constraint. They should all be the same. Otherwise, the constraint will be meaningless.)

STEP 3: Solve the Problem

As you can observe, we have a "linear" objective function. Similarly, our constraints are "linear" inequalities. The above methodology that we used for Giapetto's problem is called *Linear Programming*. A linear programming problem is an optimization problem. Now we need to find the values of X and Y, so that the objective function is maximized.

a) TI-Calculator

Any linear program with only two variables can be solved graphically. You are expected to use your TI calculators to solve Giapetto's problem graphically. What you need to find is the intersection set of all of the constraints. The intersection of all the constraints is called the *feasible region*.

First of all, run the inequalities application "*Inequalz*".

Press APPS.

Choose Inequalz.

Press ENTER.

TEXAS
INSTRUMENTS
INEQUALITY
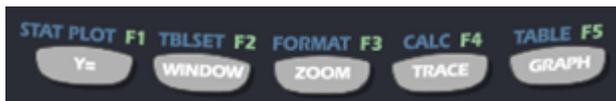GRAPHING
v 1.04
PRESS ANY KEY
©2001 TEXAS INSTRUMENTS

Press ENTER.

X= Plot1   Plot2   Plot3
\Y₁=
\Y₂=
\Y₃=
\Y₄=
\Y₅=
\Y₆=
(=)(<)(≤)(>)(≥)

At the bottom of the screen, you can see the inequality signs. In order to choose an inequality sign, use ALPHA and then one of the following buttons which corresponds to the related sign.

For example, in order to choose ≤ sign, press ALPHA and then ZOOM (i.e., F3). In order to type in the calculator, you need to rewrite your constraints so that you leave Y alone on one side of the inequality. Before starting the solution, first adjust the scales of the graph. Press WINDOW and adjust the min, max and scl values as follows.

WINDOW
 ShadeRes=3
 Xmin=-100
 Xmax=100
 Xscl=10
 Ymin=-100
 Ymax=100
↓Yscl=10

Now, write the first inequality (Constraint 1) and graph it.

X= Plot1   Plot2   Plot3
▶Y₁◘100-2X
 \Y₂█
 \Y₃=
 \Y₄=
 \Y₅=
 \Y₆=
(=)(<)(≤)(>)(≥)

Press GRAPH.
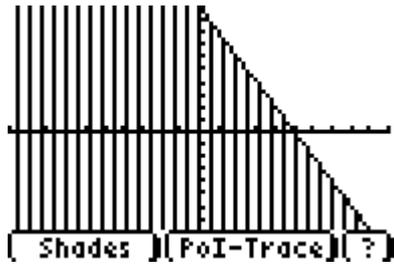
You can see the graph of the first inequality as an example. The restricted area with the first constraint can be seen as shaded. Now, type the other constraints as well.

But wait, you have inequalities with no Y value in it. So how can you type it? Look at the upper left corner of the screen.

Take the cursor to *X=* sign and press ENTER. In this screen, you can type your constraints that have no Y easily. Don't worry; your previous constraints are still saved.
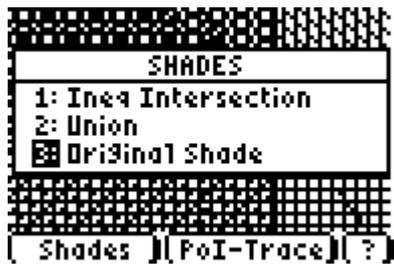

Now press GRAPH to see all of the shaded inequalities. You see a big mess of shaded region for all of the five inequalities.

In order to see the feasible region easily, you will change the shading a little bit. To choose "Shades" in the screen, press F1 (ALPHA and Y=). It gives you some options for shading.

In this question, we need to find the feasible region, i.e. the intersection of all constraints. Therefore Press 1.
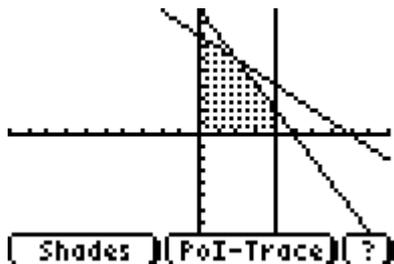

Congratulations! You have found the feasible region. A very useful theorem will help you to find the exact solution you are looking for.


Fundamental Theorem of Linear Programming: The minima and maxima of a linear function on a convex feasible region occur at the vertices of the region.

(Warning: We need to be careful whether the vertices have integer coordinates or not. If all of the vertices have integer coordinates, we need only to consider these points when finding the maximum or minimum. However, if the vertices are not all integer points and we are considering only integer points, we need to be more careful.)

As you should have seen, the feasible region has 5 vertices in this question. Now, how will you find the exact coordinates of these points?
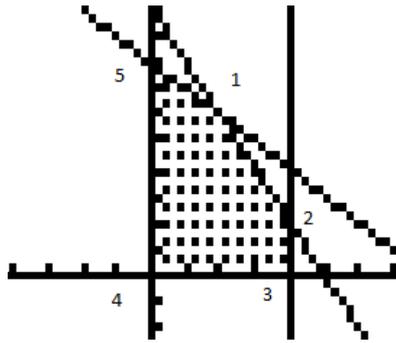
At the bottom of the screen, you see PoI-Trace button. For choosing this button, press F3 (ALPHA and ZOOM).
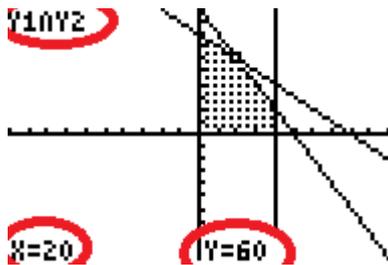
There you found one of the intersection points. It is the intersection of Y1 and Y2, which is the vertex #1. You can find the other intersection points by using the upper or lower arrows. Write down the coordinates of all five vertices below.

(X ,   Y)

Vertex #1: (,   )

Vertex #2: (,   )

Vertex #3: (,   )

Vertex #4: (,   )

Vertex #5: (,   )

Now that you have identified the possible optimal solutions, you can find the objective function value for each of them. Find the maximum value of the objective function and corresponding decision variable values.

Vertex #1: (,   ) → weekly profit =

Vertex #2: (,   ) → weekly profit =

Vertex #3: (,   ) → weekly profit =

Vertex #4: (,   ) → weekly profit =

Vertex #5: (,   ) → weekly profit =

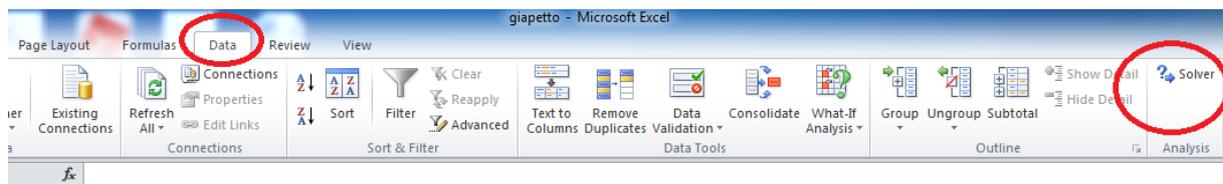∴ OPTIMAL SOLUTION: ………………………………………………………………… ■

b) MS Excel

You can solve the same question buy Using MS Excel Solver Add-in. First of all, write the constraint coefficients and objective function coefficients as follows. Spare empty cells for the decision variables (color code: yellow). Spare a cell for the maximum value of the objective function (color code: red).

|	Soldiers/week (X)	Trains /week (Y)				
Decision variables						
Constraint 1 -Finishing constraint	2	1			≤	100
Costraint 2- Carpentry constraint	1	1			≤	80
Constraint 3 -Constraint on demand	1	0			≤	40
Constraint 4 - Sign restriction	1	0			≥	0
Constraint 5 -Sign restriction	0	1			≥	0
Objective Function - Weekly profit	3	2				

For each constraint, write down the "sumproduct" of the decision variables and the constraint coefficients in column D. Fix the decision variables by using $ sign. Similarly insert the "sumproduct" of the decision variables and the objective function coefficients in cell D8.

	SUM	▾	X ✓ fx	=SUMPRODUCT($B$2:$C$2;B3:C3)		
	A	B	C	D	E	F
1		Soldiers/week	Trains /week			
2	Decision variables					
3	Constraint 1 -Finishing constraint	2	=SUMPRODUCT($B$2:$C$2;B3:C3)			
4	Costraint 2- Carpentry constraint	1	1		≤	80
5	Constraint 3 -Constraint on demand	1	0		≤	40
6	Constraint 4 - Sign restriction	1	0		≥	0
7	Constraint 5 -Sign restriction	0	1		≥	0
8	Objective Function - Weekly profit	3	2			

Now, from the "Data" tab, choose "Solver". (If you don't have excel solver, you may need to download it)

In the figure below, you see the necessary cells chosen. Choose SIMPLEX LP for solving method.
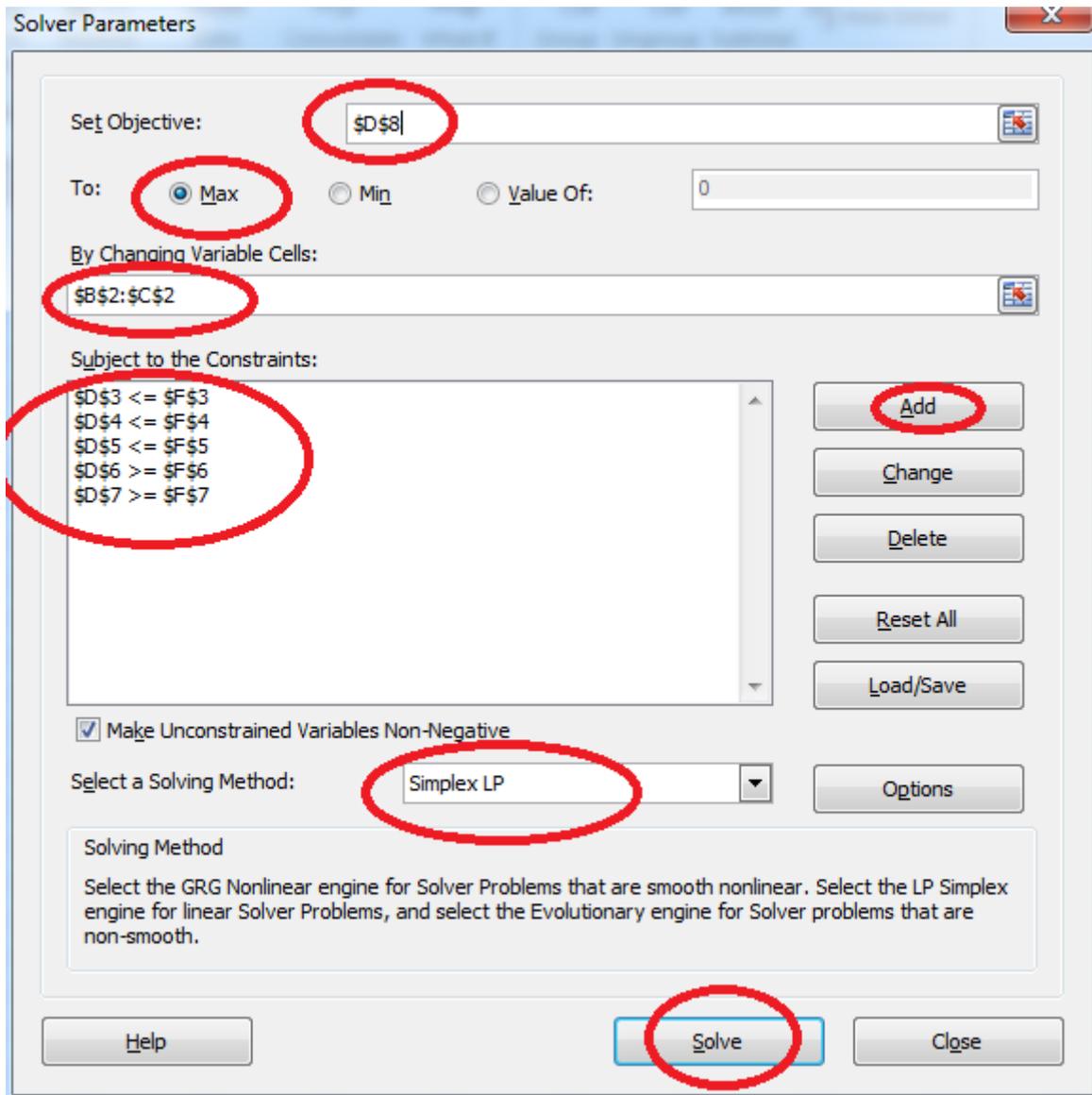
When you press SOLVE, you will see the optimal solution in the red coded cell.

STEP 4: Look Back

a) **Compare the results you found by using TI calculator and MS Excel. Are they the same? In fact, they should!**

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b) Compare the results you found in STEP 3, with your intuitive answer in STEP1-Part c. How much it differs? Why do you think so?

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REFLECTION

Operations Research is a discipline that is based on use of analytical methods *to design and operate a system* in the most efficient way. It is also closely related with *industrial engineering,* in terms of the tools that are used. For the purpose of emphasizing STEM education, I decided to use an engagement problem to one of those methods in this worksheet.

Giapetto's woodcarving problem is an introductory problem in Winston's (2004) operation research course book. By using this problem, I decided to encourage high school students about solving complex problems and to make them see how easily they can solve them by using technology. I also aimed to arouse curiosity about the mathematical facts and algorithms behind these tools. The students will make use of technological tools such as TI calculators and MS Excel to model the problem and make it less abstract. This problem can be regarded as an extension to "linear inequalities" topic both in the MoNE and the IGCSE curricula. Parallel to the mathematics curriculum guide of MoNE (2013), I aimed to render mathematical thinking "accessible" to all students from different levels of mathematical skills, by using technology.

While preparing this worksheet, I made use of TI Connect to get the screen captures. I sometimes had a dilemma about which parts to add more explanation or which parts to leave completely to the student work. I decided to hide the clues for some parts in the screenshots of different parts of the problem.